CONVECTIVE STABILITY OF A VERTICAL LAYER OF A NON-NEWTONIAN LIQUID

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We consider the convective stability of a non-Newtonian (nonlinearly viscous) liquid in a two-dimensional vertical channel. We solve a nonlinear boundary value problem concerning plane-parallel stationary convection for the case of piecewise-linear and power-law type rheological characteristics. We discuss the problem concerning the stability of equilibrium and of stationary motions.

1. In [1, 2] an experimental study was made of the origin of convection in a horizontal layer of liquid heated from below. In [1] an attempt was made, with the aid of an energy method, to estimate the critical Rayleigh number for power-law liquids. This attempt must be considered unsuccessful. If the initial viscosity is equal to zero or infinity, as is the case in power-law models, then there is no finite critical Rayleigh number defining the stability boundary relative to small perturbations; the equilibrium with respect to small perturbations is either stable for all Rayleigh numbers (pseudo-plastics) or is unstable for an arbitrarily small Rayleigh number (dilatational liquids). In judging stability, consideration must be given to finite amplitude perturbations. As was noted in [3], the energy method used in [1] was applied incorrectly.

In the case of a model with finite initial viscosity the notion of a critical Rayleigh number is justified. In this case, as was shown in [2], measurement of the critical temperature gradient yields a sufficiently exact method of determining the initial viscosity.

Let a two-dimensional infinite layer of a non-Newtonian liquid, bounded by the vertical planes $x = \pm h$, be heated from below. We consider a stationary plane-parallel convective motion for which only the vertical velocity component is non-zero. In this case (the z axis is directed vertically upwards)

$$v_x = v_y = 0, \qquad v_z = v(x) \tag{1.1}$$

and the distributions of temperature T, stress τ, and pressure p have the form

$$T = -Az + \theta(x), \quad \tau = \tau(x), \quad p = p(z) \tag{1.2}$$

Here A is the constant vertical temperature gradient corresponding to mechanical equilibrium.

From the convection equations, written in the Boussinesq approximation, we obtain equations for the functions v, θ, τ, and p

$$\begin{gathered}\frac{1}{\rho}\tau' + g\beta\theta = -\frac{1}{\rho}\frac{dp}{dz} + g\beta Az = C \\ \chi\theta'' + Av = 0\end{gathered} \tag{1.3}$$

Here ρ is the average density, g is the gravitational acceleration, β and χ are, respectively, the thermal expansion and thermal diffusivity coefficients, and C is a separation of variables constant. The prime indicates differentiation with respect to x. To Eqs. (1.3) must be added the rheological relation connecting the shear stress with the velocity gradient

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$$\tau = \tau(v') \tag{1.4}$$

Along the channel boundaries the velocity vanishes and a linear temperature distribution is maintained along the vertical; in addition, we assume that the condition for the convective flow to be closed is satisfied, which means that the outflow through an arbitrary section is equal to zero. Thus we have

$$v(\pm h) = \theta(\pm h) = 0, \quad \int_{-h}^{h} v\,dx = 0 \tag{1.5}$$

Further, we consider a stationary motion, corresponding to the lower level of instability of equilibrium. In this motion the velocity profiles v and the temperatures θ are odd functions relative to the middle of the layer, x=0; the separation constant C = 0, and the closure condition is satisfied.

The problem we have formulated has a trivial solution ($v = \theta = \tau = 0$), corresponding to equilibrium (stable or unstable, depending on the relationship among the parameters) of the liquid heated from below. Under certain conditions nontrivial solutions are possible; we consider these in what follows.

In the case of a Newtonian liquid the relation (1.4) is linear and nontrivial solutions exist only for certain values of the temperature gradient (Rayleigh number), these values forming a discrete spectrum (see [4]). These characteristic values of the gradient are, at the same time, critical values from the point of view of the stability of equilibrium. The amplitude of the characteristic motions turns out to be indeterminate. In the case considered here of a non-Newtonian liquid the dependence of the stress (1.4) on the velocity gradient is nonlinear. This leads to qualitative differences. Stationary motions exist, not at isolated points of the spectrum, but at all the points of some interval of values of the temperature gradient (Rayleigh number). The amplitude of the motions, by virtue of the nonlinearity of the boundary value problem, is found to be determinate.

2. We consider a liquid with a piecewise-linear rheological characteristic (Fig. 1). The dependence $\tau(v')$ is of the form

$$\tau = \begin{cases} \mu_1 v', & -\tau_0 \leqslant \tau \leqslant \tau_0 \\ \tau_0 + \mu_2(v' - v_0'), & \tau \geqslant \tau_0 \\ -\tau_0 + \mu_2(v' + v_0'), & \tau \leqslant -\tau_0 \end{cases} \tag{2.1}$$

The characteristic (2.1) contains three independent parameters: an initial viscosity μ_1 and a limiting viscosity μ_2; also, a limiting tangential stress τ_0 ($v_0' = \tau_0/\mu_1$). The viscosity is constant on each of the stress intervals in the functional dependence (2.1) and changes by a jump at $\tau = \pm\tau_0$. A piecewise-linear characteristic can be considered as an approximation for describing nonlinearly viscous liquids with finite values for the initial and the limiting viscosity. From the relation (2.1) we can obtain the relations corresponding to pseudo-plastic ($\mu_1 > \mu_2$) and dilatational ($\mu_1 < \mu_2$) behavior (curves 1 and 2 in Fig. 1). As limiting cases, we have a Bingham liquid ($\mu_1 = \infty$, curve 3) and the limiting case of a dilational liquid with zero initial viscosity ($\mu_1 = 0$, curve 4). For a Newtonian liquid we have $\mu_1 = \mu_2$.

Keeping in mind the characteristic (2.1), we write the equations of plane-parallel convection in dimensionless form. We introduce the following units: distance h, velocity χ/h, temperature Ah, and stress $\mu_2\chi/h^2$. We can then write Eqs. (1.3), the rheological relation (2.1), and the boundary conditions (1.5) in the form

$$\begin{gathered} \tau' + R\theta = 0, \quad \theta'' + v = 0 \\ \tau = \begin{cases} \mu v', & |\tau| \leqslant B \\ B(1 - 1/\mu)\operatorname{sign} v' + v', & |\tau| \geqslant B \end{cases} \\ v(\pm 1) = \theta(\pm 1) = 0, \quad \int_{-1}^{1} v\,dx = 0 \end{gathered} \tag{2.2}$$

The boundary value problem (2.2) contains three dimensionless parameters: the Rayleigh number R, defined for the viscosity μ_2; the dimensionless limiting stress B; and the ratio of viscosities, μ. Thus,

$$R = \frac{\rho g \beta A h^4}{\mu_2 \chi}, \quad B = \frac{\tau_0 h^2}{\mu_2 \chi}, \quad \mu = \frac{\mu_1}{\mu_2} \tag{2.3}$$

The basic stationary motion corresponds to an odd solution of the problem (2.2) relative to the point $x=0$. In view of the oddness of the solution, it is sufficient to consider the region $0 \le x \le 1$. In this region we can single out a middle zone $a \le x \le b$ (subscript 0) with a small velocity gradient and viscosity μ_1; the zone $0 \le x \le a$ (subscript 1) in which $\tau > \tau_0$, and in which the velocity gradient is positive and in magnitude greater than the limiting value; and the zone $b \le x \le 1$ (subscript 2) with a negative velocity gradient and $\tau < -\tau_0$ (in zones 1 and 2 we have viscous flow with viscosity μ_2).

For each of these zones we can write the general solution of Eqs. (2.2). Taking into account the vanishing of the velocity and the temperature at the points $x=0$ and $x=1$, we have

$$\begin{aligned}
v_1 &= D_1 \sin rx + D_2 \operatorname{sh} rx \\
\theta_1 &= r^{-2} (D_1 \sin rx - D_2 \operatorname{sh} rx) \\
\tau_1 &= B(1-\mu^{-1}) + r(D_1 \cos rx + D_2 \operatorname{ch} rx) \\
v_2 &= C_1 \sin r(1-x) + C_2 \operatorname{sh} r(1-x) \\
\theta_2 &= r^{-2}[C_1 \sin r(1-x) - C_2 \operatorname{sh} r(1-x)] \\
\tau_2 &= -B(1-\mu^{-1}) - r[C_1 \cos r(1-x) + C_2 \operatorname{ch} r(1-x)] \\
v_0 &= E_1 \sin sx + E_2 \cos sx + E_3 \operatorname{sh} sx + E_4 \operatorname{ch} sx \\
\theta_0 &= s^{-2}(E_1 \sin sx + E_2 \cos sx - E_3 \operatorname{sh} sx - E_4 \operatorname{ch} sx) \\
\tau_0 &= \mu s (E_1 \cos sx - E_2 \sin sx + E_3 \operatorname{ch} sx + E_4 \operatorname{sh} sx)
\end{aligned} \tag{2.4}$$

where $r=R^{1/4}$, $s=(R/\mu)^{1/4}$.

For the determination of the eight arbitrary constants and the unknown parameters a and b, defining the locations of the viscous zone boundaries, we have matching conditions at the points a and b (continuity of the velocity, temperature, thermal flow, and stress) and also the conditions defining the location of the zone boundaries

$$\begin{aligned}
&v_1(a) = v_0(a), \quad \theta_1(a) = \theta_0(a), \quad \theta_1'(a) = \theta_0'(a) \\
&\tau_1(a) = \tau_0(a), \quad \tau_0(a) = B \\
&v_0(b) = v_2(b), \quad \theta_0(b) = \theta_2(b), \quad \theta_0'(b) = \theta_2'(b) \\
&\tau_0(b) = \tau_2(b), \quad \tau_0(b) = -B
\end{aligned} \tag{2.5}$$

From an analysis of the relations (2.5) it follows that all the profiles are symmetric relative to the point $x=1/2$; in particular, $a=1-b$. The constants appearing in Eqs. (2.4) are equal to

$$D_1 = \frac{B}{\mu r\delta \cos ra}[2\mu^{3/4} + (1+\sqrt{\mu}) \operatorname{th} ra \operatorname{th} \varphi + (1-\sqrt{\mu}) \operatorname{th} ra \operatorname{tg} \varphi]$$

$$D_2 = \frac{B}{\mu r\delta \operatorname{ch} ra}[-2\mu^{3/4} + (1+\sqrt{\mu}) \operatorname{tg} ra \operatorname{tg} \varphi + (1-\sqrt{\mu}) \operatorname{tg} ra \operatorname{th} \varphi]$$

$$E_1 = \frac{B \sin s/2}{\mu r\delta \cos \varphi}[2 \operatorname{tg} ra \operatorname{th} ra \operatorname{th} \varphi + \mu^{1/4}(1+\sqrt{\mu}) \operatorname{tg} ra + \mu^{1/4}(1-\sqrt{\mu}) \operatorname{th} ra]$$

$$E_3 = \frac{B \operatorname{sh} s/2}{\mu r\delta \operatorname{ch} \varphi}[-2 \operatorname{tg} ra \operatorname{th} ra \operatorname{tg} \varphi + \mu^{1/4}(1+\sqrt{\mu}) \operatorname{th} ra + \mu^{1/4}(1-\sqrt{\mu}) \operatorname{tg} ra]$$

$$E_2 = \operatorname{ctg}\frac{s}{2} E_1, \quad E_4 = -\operatorname{cth}\frac{s}{2} E_3, \quad C_1 = D_1, \quad C_2 = D_2$$

$$\delta = (1+\sqrt{\mu})(\operatorname{tg} ra \operatorname{tg} \varphi + \operatorname{th} ra \operatorname{th} \varphi) + (1-\sqrt{\mu})(\operatorname{th} ra \operatorname{tg} \varphi + \operatorname{tg} ra \operatorname{th} \varphi)$$

$$\varphi = s(1/2 - a)$$

The relation defining the parameter a as a function of R and μ has the form

$$\begin{aligned}
&(1+\sqrt{\mu})^2(\operatorname{tg} ra \operatorname{tg}\varphi - \operatorname{th} ra \operatorname{th} \varphi) + (1-\sqrt{\mu})^2(\operatorname{tg} ra \operatorname{th}\varphi - \\
&- \operatorname{th} ra \operatorname{tg} \varphi) + 4\mu^{1/4} \operatorname{tg} \varphi \operatorname{th} \varphi \operatorname{tg} ra \operatorname{th} ra - 4\mu^{3/4} = 0
\end{aligned} \tag{2.6}$$

We give the expression for the maximum flow velocity

$$\begin{aligned}
v_m = v_0\left(\frac{1}{2}\right) = &\frac{B}{\mu r\delta \cos\varphi \operatorname{ch}\varphi}[2 \operatorname{tg} ra \operatorname{th} ra (\operatorname{sh}\varphi + \sin\varphi) + \\
&+ \mu^{1/4}(1+\sqrt{\mu})(\operatorname{tg} ra \operatorname{ch}\varphi - \operatorname{th} ra \cos\varphi) + \mu^{1/4}(1-\sqrt{\mu})(\operatorname{th} ra \operatorname{ch}\varphi - \operatorname{tg} ra \cos\varphi)]
\end{aligned} \tag{2.7}$$

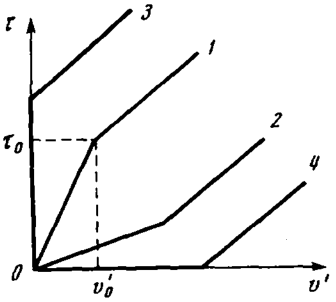

Fig. 1

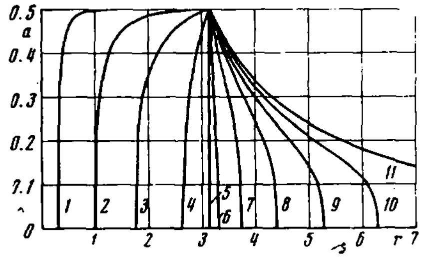

Fig. 2

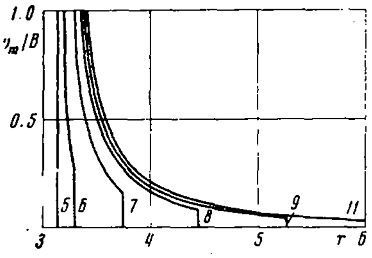

Fig. 3

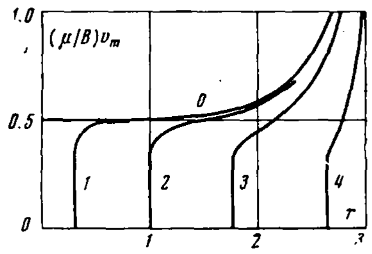

Fig. 4

3. The results obtained in solving numerically the transcendental equation (2.6) are shown in Fig. 2, which presents the family of curves $a(r)$ for various values of the viscosity ratio μ. Figures 3 and 4 show the dependence of the amplitude of the velocity on r. The numbering of the curves, namely, 0, 1, 2, ..., 11, corresponds to values of $\mu = 0,\ 10^{-4},\ 10^{-2},\ 0.1,\ 0.5,\ 1,\ 1.2,\ 2,\ 4,\ 8,\ 16,\ \infty$.

We consider the pseudo-plastic liquids ($\mu = \mu_1/\mu_2 > 1$). Analysis of the relations (2.6), (2.7) shows that in this case a nontrivial solution, corresponding to plane-parallel convection, exists in the restricted interval of Rayleigh numbers $\pi^4 < R < \mu\pi^4$, i.e., $\pi < r < \mu^{1/4}\pi$.

The point $r = \mu^{1/4}\pi$ is a critical point in the sense of the stability of equilibrium relative to small perturbations. Small perturbations of equilibrium correspond to small v', i.e., to small stresses τ. Such perturbations evolve in the same way as in a Newtonian liquid with viscosity coefficient μ_1 (the initial part of the rheological curve). An equilibrium crisis relative to these perturbations is determined by the condition $\rho g\beta A h^4/\mu_1\chi = \pi^4$. Changing over to the parameters R and μ, we write this condition in the form $R/\mu = \pi^4$, i.e., $r = \mu^{1/4}\pi$. For $r < \mu^{1/4}\pi$ the equilibrium is stable relative to small perturbations; for $r > \mu^{1/4}\pi$ the equilibrium is unstable .

In the region $\pi < r < \mu^{1/4}\pi$, in addition to the equilibrium solution we also have a nontrivial finite amplitude solution with determinate values of the parameters a and v_m. With increasing r in this region the parameter a, as is evident from Fig. 2, decreases monotonically from 1/2 to 0. The amplitude v_m is proportional to the dimensionless limiting stress B. With an increase of r in the region $\pi < r < \mu^{1/2}\pi$ the amplitude v_m decreases from infinity to some finite value $B/\pi\mu$; at the point $r = \mu^{1/4}\pi$ the amplitude undergoes a jump.

We can obtain asymptotic expressions for a and v_m from Eqs. (2.6) and (2.7). Close to the lower critical point $r = \pi$

$$a = \frac{1}{2} - \frac{\mu}{\pi(\mu-1)}(r-\pi) + \dots, \qquad v_m = \frac{B(\mu-1)}{\pi\mu(r-\pi)} + \dots \tag{3.1}$$

Close to the upper critical point $r = \mu^{1/4}\pi$

$$a \cong \left[\frac{6}{\pi^2(\mu-1)}\left(1 - \frac{r}{\pi\mu^{1/4}}\right)\right]^{1/2} \tag{3.2}$$

The finite amplitude stationary solution found here is unstable. For $r < \mu^{1/4}\pi$ the equilibrium is stable relative to small perturbations. If we insert into the equilibrium a perturbation of finite amplitude, entering sufficiently far from the limits of the initial portion of the rheological curve, then layers are formed in the flow with the smaller viscosity μ_2. This means a lowering of the effective viscosity of the system, as a result of which a perturbation of sufficiently large amplitude is found to be increasing. If the amplitude is less than v_m, defined by the expression (2.7), the perturbation decays; if the amplitude exceeds v_m, the perturbation increases without bound.

Thus, in the case $\mu > 1$ the equilibrium is stable with respect to small perturbations for $r < \mu^{1/4}\pi$ and unstable for $r > \mu^{1/4}\pi$. In the region $\pi < r < \mu^{1/4}\pi$ the equilibrium is unstable relative to finite perturbations of amplitude greater than v_m ("rigid" perturbation). For $r < \pi$ the equilibrium is stable with respect to perturbations of arbitrary amplitude.

When $\mu > 1$, there are no stable stationary motions of finite amplitude. This is connected with the specific geometry of the problem in question, namely, plane-parallel motions in an infinite layer. In this case the nonlinear convective terms in the equations of motion and heat conduction, i.e., the terms $(v\nabla)\, v$ and $v\nabla\theta$, vanish identically, and the nonlinearity associated with the rheological curve $\tau(v')$ is of a destabilizing nature for $\mu > 1$.

With an increase in μ, the Rayleigh number, defining the upper critical point, increases and tends towards infinity for $\mu \to \infty$. This limiting case corresponds to a Bingham liquid ($\mu_1 = \infty$, μ_2 finite). The equilibrium is stable relative to small perturbations for all R. Only a rigid perturbation of the convection is possible for $R > \pi^4$; the critical amplitude is given by the value

$$v_m = \frac{24B}{r^3(1-2a)^2}\,\frac{1}{3(\operatorname{tg} ra + \operatorname{th} ra) + r(1-2a)}$$

The boundary a of the viscous and plastic zones of flow is obtained from the equation

$$r(1-2a)(\operatorname{tg} ra - \operatorname{th} ra) - 4 = 0$$

These expressions are obtained from Eqs. (2.6) and (2.7) in the limit as $\mu \to \infty$ and they coincide with the results obtained in [5].

4. We consider now the case of a dilatational liquid ($\mu = \mu_1/\mu_2 < 1$). The dependence of the zone boundary a and the stationary amplitude v_m on the Rayleigh number is shown in Figs. 2 and 4. Just as in the pseudoplastic case, the point $r = \mu^{1/4}\pi$ is a critical point in the sense of the stability of equilibrium relative to small perturbations; for $r > \mu^{1/4}\pi$ the equilibrium is unstable. Stationary motion of finite amplitude exists in the interval $\mu^{1/4}\pi < r < \pi$. At the point $r = \mu^{1/4}\pi$, through a jump, there arises a stationary motion with the amplitude $v_m = B/\pi\mu$, and as r increases, the amplitude v_m increases monotonically, tending towards infinity as $r \to \pi$. The asymptotic expressions (3.1) and (3.2) continue to hold even in the case $\mu < 1$.

In the case $\mu < 1$ a stationary plane-parallel motion of finite amplitude in the region $\mu^{1/4}\pi < r < \pi$ is stable. Small perturbations of equilibrium in this region grow, and, upon the attainment of a limiting stress in the flow, layers of large viscosity arise. An increase in the effective viscosity of the system leads to a stabilization of perturbations and to the establishment of a stationary amplitude v_m. For $r > \pi$, there can be no stable stationary plane-parallel motion, since for such Rayleigh numbers small perturbations of equilibrium increase without bound, even in the case of a Newtonian liquid with a large viscosity μ_2.

The case of a limiting dilatational characteristic (curve 4 in Fig. 1) is obtained for $\mu \to 0$ and $B \to 0$ (the ratio B/μ, being the dimensionless value of the limiting velocity gradient v_0', stays finite). From the general expressions (2.6) and (2.7) we obtain

$$a = \frac{1}{2}, \qquad v_m = \frac{B}{2\mu r}\left(\operatorname{tg}\frac{r}{2} + \operatorname{th}\frac{r}{2}\right)$$

In this limiting case the equilibrium is unstable for an arbitrarily small temperature gradient; a stable stationary motion exists in the region $0 < r < \pi$.

5. We consider a liquid whose behavior corresponds to the rheological power-law

$$\tau = k|v'|^n \operatorname{sign} v' \tag{5.1}$$

where n is an exponent and k is the coefficient of consistency. To determine the stationary plane-parallel motion in a layer of power-law liquid, the problem (1.3)-(1.5) must be solved with the rheological law (5.1).

We choose the following as our units: distance h, velocity $(\chi/h)R^{1/(n-1)}$, and temperature $AhR^{1/(n-1)}$, where $R=\rho g\beta Ah^{2n+2}/k\chi^n$ is the modified Rayleigh number.

In dimensionless variables the problem assumes the form

$$[|v'|^n \operatorname{sign} v']' + \theta = 0, \quad \theta'' + v = 0]$$
$$v(\pm 1) = \theta(\pm 1) = 0, \quad \int_{-1}^{1} v dx = 0 \tag{5.2}$$

This problem does not contain the Rayleigh number R; the exponent n is the only parameter determining the solution.

For the purpose of obtaining a numerical solution the equations were reduced to a system of four first order equations. This system was integrated from the point x=0 to x=1 (we have in mind here an odd solution). The missing boundary conditions at the left end were determined by inspection until the boundary conditions at the right end could be satisfied with sufficient accuracy. In the numerical integration we used the Runge–Kutta and the "predictor-corrector" methods. As a result we obtained the distribution v(x) and θ (x) for various values of n. In the case n < 1 we have a plateau on the velocity profiles which is characteristic of pseudo-plastics; for n > 1 corners are formed alongside the point x=1/2. These profiles are close to those given in [6] for the case of the flow of a power-law liquid in a channel heated from the side.

Numerical integration of the system (5.2) enables us to determine the maximum dimensionless velocity depending on n. The calculations lead to the empirical dependence $v_m \sim \pi^{-4/(n-1)}$. Taking note of the units chosen, we can represent the maximum dimensional velocity in the form

$$v_m = c\frac{\chi}{h}\left(\frac{R}{R_0}\right)^{1/(n-1)}$$

Here $R_0 = \pi^4$ is the critical Rayleigh number in the case of a Newtonian liquid (n=1), and the coefficient c is a slowly varying function of n. In the interval $1/3 \le n \le 2$ c we have, with sufficient accuracy, c=0.34.

In the case n < 1 (pseudo-plastics) the amplitude v_m of the velocity of the stationary motion decreases monotonically from infinity to zero as R increases; as n is varied we obtain a family of hyperbolas of various orders (the analog of the curves in Fig. 3). This case differs from the case of a piecewise-linear characteristic in that the equilibrium is stable with respect to small perturbations for all R. This is explained by the infinite initial viscosity of a power-law pseudo-plastic. For all R, however, we have instability with respect to finite perturbations exceeding the magnitude v_m. A stationary mode with the velocity v_m, as in the case of a piecewise-linear characteristic, is unstable.

For n > 1 (dilatational liquids), equilibrium is unstable with respect to small perturbations for all R, beginning with arbitrarily small perturbations (zero initial viscosity). The stationary plane-parallel mode with the velocity v_m is stable. The velocity v_m increases monotonically from zero depending upon the increase in R; the curves $v_m(R)$ for n > 1 constitute a family of parabolas of various orders (the analog of the curves in Fig. 4).

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